Worksheet n°1 : mesh, base-flow, global modes, adjoint global modes

Codes may be downloaded on: http://denissipp.free.fr/teaching/Codes/NumStud.zip

0/ Very short reminder on finite elements

Let us solve the following problem:

$$u-(\partial_{xx}u+\partial_{yy}u)=f$$

$$u=d \text{ on } \Gamma_d$$

$$au+b\partial_n u=c \text{ on } \Gamma_m$$

We consider test functions $\breve{u}$ satisfying $\breve{u}=0$ on Γ_d. After multiplying the governing equation by the test-function, we take an integral over the complete domain:

$$\iint \breve{u}\big(u-(\partial_{xx}u+\partial_{yy}u\big)dxdy=\iint \breve{u}f\,dxdy$$

Integrating by parts, we obtain:

$$\iint \big(\breve{u}u+\partial_x\breve{u}\partial_x u+\partial_y\breve{u}\partial_y u\big)dxdy-\int \big(\breve{u}n_x\partial_x u+\breve{u}n_y\partial_y u\big)ds=\iint \breve{u}f\,dxdy$$

The boundary term is zero on Γ_d because of $\breve{u}=0$. Therefore, taking into account the boundary condition on Γ_m, we have:

$$\iint \big(\breve{u}u+\partial_x\breve{u}\partial_x u+\partial_y\breve{u}\partial_y u\big)dxdy-\int_{\Gamma_m}\breve{u}\left(c-\frac{a}{b}u\right)ds=\iint \breve{u}f\,dxdy$$

Rearranging:

$$\iint \big(\breve{u}u+\partial_x\breve{u}\partial_x u+\partial_y\breve{u}\partial_y u\big)dxdy+\int_{\Gamma_m}\frac{a}{b}\breve{u}u\,ds=\iint \breve{u}f\,dxdy+\int_{\Gamma_m}\breve{u}c\,ds$$

Using for example P2 elements for u and $\breve{u}$, we obtain the following discretized form (taking into account that $u=d$ on Γ_d):

$$Au=b$$

1/ Generate mesh

In folder Mesh:

FreeFem++ mesh.edp

2/ Base-flow

The base-flow is solution of the following non-linear equation:

$$\frac{1}{2}\mathcal{N}(w_0,w_0)+\mathcal{L}w_0=0,\quad \mathcal{N}(w_1,w_2)=\begin{pmatrix}u_1\cdot\nabla u_2+u_2\cdot\nabla u_1\\0\end{pmatrix},\quad \mathcal{L}=\begin{pmatrix}-\nu\Delta() & \nabla()\\-\nabla\cdot() & 0\end{pmatrix}$$

with the following boundary conditions:

$$(u_0=1,v_0=0)\ \text{ on } \Gamma_{in}$$

$$(u_0=0,v_0=0)\ \text{ on } \Gamma_{wall}$$

$$\big(-p_0n_x+\nu(n_x\partial_x u_0+n_y\partial_y u_0)=0,-p_0n_y+\nu(n_x\partial_x v_0+n_y\partial_y v_0)=0\big)\ \text{ on } \Gamma_{out}$$

$$\big(\partial_y u_0=0,v_0=0\big)\ \text{ on } \Gamma_{lat}$$

The Newton iteration is based on successive solutions of:

$$(\mathcal{N}_w+\mathcal{L})\delta w=-\frac{1}{2}\mathcal{N}(w,w)-\mathcal{L}w \text{ where } \mathcal{N}_w\delta w=\begin{pmatrix}\delta u\cdot\nabla u+u\cdot\nabla\delta u\\0\end{pmatrix}$$

with boundary conditions such that $w+\delta w$ satisfy the above mentioned boundary conditions.

Hence:

$$\begin{aligned}\delta u\partial_x u+\delta v\partial_y u+u\partial_x\delta u+v\partial_y\delta u+\partial_x\delta p-\nu\big(\partial_{xx}\delta u+\partial_{yy}\delta u\big)\\=-u\partial_x u-v\partial_y u-\partial_x p+\nu\big(\partial_{xx}u+\partial_{yy}u\big)\end{aligned}$$

$$\begin{aligned}\delta u\partial_x v+\delta v\partial_y v+u\partial_x\delta v+v\partial_y\delta v+\partial_y\delta p-\nu\big(\partial_{xx}\delta v+\partial_{yy}\delta v\big)\\=-u\partial_x v-v\partial_y v-\partial_y p+\nu\big(\partial_{xx}v+\partial_{yy}v\big)\end{aligned}$$

$$-\partial_x\delta u-\partial_y\delta v=\partial_x u+\partial_y v$$

with:

$$(\delta u=1-u,\delta v=-v) \text{ on } \Gamma_{in}$$

$$(\delta u=-u,\delta v=-v)\ \text{ on } \Gamma_{wall}$$

$$\begin{aligned}\big(-\delta p n_x+\nu\big(n_x\partial_x\delta u+n_y\partial_y\delta u\big)=pn_x-\nu\big(n_x\partial_x u+n_y\partial_y u\big),-\delta p n_y+\nu\big(n_x\partial_x\delta v+n_y\partial_y\delta v\big)\\=pn_y-\nu\big(n_x\partial_x v+n_y\partial_y v\big) \text{ on } \Gamma_{out}\big)\end{aligned}$$

$$(\partial_y\delta u=-\partial_y u,\delta v=-v\ ) \text{ on } \Gamma_{lat}$$

Show that the weak form of these equations is (with $\breve{w}$ as the test-function satisfying $\breve{u}=\breve{v}=0$ on Γ_{in} and Γ_{wall} and $\breve{v}=0$ on Γ_{lat})

$$\begin{aligned}\iint \big(\breve{u}\big(\delta u\partial_x u+\delta v\partial_y u+u\partial_x\delta u+v\partial_y\delta u\big)+\breve{v}\big(\delta u\partial_x v+\delta v\partial_y v+u\partial_x\delta v+v\partial_y\delta v\big)-\delta p(\partial_x\breve{u}+\partial_y\breve{v})\\+\nu\big(\partial_x\breve{u}\partial_x\delta u+\partial_y\breve{u}\partial_y\delta u+\partial_x\breve{v}\partial_x\delta v+\partial_y\breve{v}\partial_y\delta v\big)-\breve{p}(\partial_x\delta u\\+\partial_y\delta v)\ \big)dxdy\ =\iint\big(-\breve{u}\big(u\partial_x u+v\partial_y u\big)-\breve{v}\big(u\partial_x v+v\partial_y v\big)+p\big(\partial_x\breve{u}+\partial_y\breve{v}\big)\\-\nu\big(\partial_x\breve{u}\partial_x u+\partial_y\breve{u}\partial_y u+\partial_x\breve{v}\partial_x v+\partial_y\breve{v}\partial_y v\big)+\breve{p}(\partial_x u+\partial_y v)\big)dxdy\end{aligned}$$

After discretization (taking into account all the Dirichlet boundary-conditions), we obtain:

$$A\delta w=b$$

In folder BF:

vi param.txt	// target Reynolds number, here Re=100
FreeFem++ init.edp	// generate initial guess solution, here zero flowfield
FreeFem++ newton.edp	// compute base-flow
FreeFem++ plotUvvp.edp	// show base-flow at Re=100

3/ Global modes

The global modes are the structures such that:

$$\lambda \mathcal{B}\widehat{w} + \left(\mathcal{N}_{w_0} + \mathcal{L}\right)\widehat{w} = 0, \quad \mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

where $\left(\mathcal{N}_{w_0} + \mathcal{L}\right)$ is the linearized Navier-Stokes operator:

$$\left(\mathcal{N}_{w_0} + \mathcal{L}\right)\widehat{w} = \begin{pmatrix} \hat{u}\partial_x u_0 + \hat{v}\partial_y u_0 + u_0\partial_x \hat{u} + v_0 \partial_y \hat{u} + \partial_x \hat{p} - \nu\left(\partial_{xx}\hat{u} + \partial_{yy}\hat{u}\right) \\ \hat{u}\partial_x v_0 + \hat{v}\partial_y v_0 + u_0\partial_x \hat{v} + v_0 \partial_y \hat{v} + \partial_y \hat{p} - \nu\left(\partial_{xx}\hat{v} + \partial_{yy}\hat{v}\right) \\ -\left(\partial_x \hat{u} + \partial_y \hat{v}\right) \end{pmatrix}$$

$\left(\mathcal{N}_{w_0} + \mathcal{L}\right)$ acts on a subspace of functions $\widehat{w}$ satisfying the following boundary conditions $\widehat{(*)}$

$$\left(\hat{u} = 0, \hat{v} = 0\right) \text{ on } \Gamma_{in} \text{ and } \Gamma_{wall}$$

$$\left(-\hat{p}n_x + \nu\left(n_x\partial_x \hat{u} + n_y \partial_y \hat{u}\right) = 0, -\hat{p}n_y + \nu\left(n_x \partial_x \hat{v} + n_y \partial_y \hat{v}\right) = 0\right) \text{ on } \Gamma_{out}$$

$$\left(\partial_y \hat{u} = 0, \hat{v} = 0\right) \text{ on } \Gamma_{lat}$$

Show that the weak form of these equations is (with $\breve{w}$ as the test-function satisfying $\breve{u} = \breve{v} = 0$ on Γ_{in} and Γ_{wall} and $\breve{v} = 0$ on Γ_{lat}):

$$\iint \Big(\breve{u}\left(-\hat{u}\partial_x u_0 - \hat{v}\partial_y u_0 - u_0 \partial_x \hat{u} - v_0 \partial_y \hat{u}\right) + \left(\partial_x \breve{u}\right)\hat{p} - \nu\left(\partial_x \breve{u}\partial_x \hat{u} + \partial_y \breve{u} \partial_y \hat{u}\right) + \breve{v}\left(-\hat{u}\partial_x v_0 - \hat{v}\partial_y v_0 - u_0 \partial_x \hat{v} - v_0 \partial_y \hat{v}\right)$$
$$+ \left(\partial_y \breve{v}\right)\hat{p} - \nu\left(\partial_x \breve{v}\partial_x \hat{v} + \partial_y \breve{v}\partial_y \hat{v}\right) + \breve{p}\left(\partial_x \hat{u} + \partial_y \hat{v}\right)\Big)\, dxdy = \lambda \iint \left(\breve{u}\hat{u} + \breve{v}\hat{v}\right) dxdy$$

With a finite element-discretization:

$$A\widehat{w} = \lambda B \widehat{w}$$

In folder Eigs:

FreeFem++ eigen.edp:

4/ Definition of adjoint operator.

The adjoint operator $\left(\widetilde{\mathcal{N}}_{w_0} + \tilde{\mathcal{L}}\right)$ is the operator satisfying for all $\widehat{w}$ and $\widetilde{w}$ the following relations:

$$< \widetilde{w}, \left(\mathcal{N}_{w_0} + \mathcal{L}\right)\widehat{w} > = < \left(\widetilde{\mathcal{N}}_{w_0} + \tilde{\mathcal{L}}\right)\widetilde{w}, \widehat{w} >$$

Here $\widehat{w}$ is in the subspace satisfying the boundary conditions $\widehat{(*)}$.

Determine the adjoint operator $\left(\widetilde{\mathcal{N}}_{w_0} + \tilde{\mathcal{L}}\right)$and the boundary conditions $\widetilde{(*)}$ that $\widetilde{w}$ satisfies.

Solution:

$$\left(\widetilde{\mathcal{N}}_{w_0} + \tilde{\mathcal{L}}\right)\widetilde{w} = \begin{pmatrix} -u_0\partial_x \tilde{u} - v_0 \partial_y \tilde{u} + \tilde{u}\partial_x u_0 + \tilde{v}\partial_x v_0 + \partial_x \tilde{p} - \nu\left(\partial_{xx}\tilde{u} + \partial_{yy}\tilde{u}\right) \\ -u_0 \partial_x \tilde{v} - v_0 \partial_y \tilde{v} + \tilde{u}\partial_y u_0 + \tilde{v}\partial_y v_0 + \partial_y \tilde{p} - \nu\left(\partial_{xx}\tilde{v} + \partial_{yy}\tilde{v}\right) \\ -\left(\partial_x \tilde{u} + \partial_y \tilde{v}\right) \end{pmatrix}$$

$$\left(\tilde{u} = 0, \tilde{v} = 0\right) \text{ on } \Gamma_{in} \text{ and } \Gamma_{wall}$$

$$\Big(-\tilde{p}n_x + \nu\partial_x \tilde{u} n_x + \nu \partial_y \tilde{u} n_y = -\tilde{u}u_0 n_x - \tilde{u}v_0 n_y, -\tilde{p}n_y + \nu \partial_x \tilde{v} n_x + \nu \partial_y \tilde{v} n_y$$
$$= -\tilde{v}u_0 n_x - \tilde{v}v_0 n_y\Big) \text{ on } \Gamma_{out}$$

$$\left(\partial_y \tilde{u} = 0, \tilde{v} = 0\right) \text{ on } \Gamma_{lat}$$

5/ The adjoint global modes are solution of the following eigen-problem :

$$\lambda \mathcal{B}\widetilde{w} + \left(\widetilde{\mathcal{N}_{w_0}} + \tilde{\mathcal{L}}\right)\widetilde{w} = 0$$

with the above mentioned boundary conditions.

Show that the weak form of these equations is:

$$\iint \Big(\breve{u}\left(u_0\partial_x \tilde{u} + v_0 \partial_y \tilde{u} - \tilde{u}\partial_x u_0 - \tilde{v}\partial_x v_0\right) + \left(\partial_x \breve{u}\right)\tilde{p} - \nu\left(\partial_x \breve{u}\partial_x \tilde{u} + \partial_y \breve{u}\partial_y \tilde{u}\right) + \breve{v}\left(u_0 \partial_x \tilde{v} + v_0 \partial_y \tilde{v} - \tilde{u}\partial_y u_0 - \tilde{v}\partial_y v_0\right)$$
$$+ \left(\partial_y \breve{v}\right)\tilde{p} - \nu\left(\partial_x \breve{v}\partial_x \tilde{v} + \partial_y \breve{v}\partial_y \tilde{v}\right) + \breve{p}\left(\partial_x \tilde{u} + \partial_y \tilde{v}\right)\Big)\, dxdy$$
$$- \int_{\Gamma_{out}} \breve{u}\left(\tilde{u}u_0 n_x + \tilde{u}v_0 n_y\right) ds - \int_{\Gamma_{out}} \breve{v}\left(\tilde{v}u_0 n_x + \tilde{v}v_0 n_y\right) ds = \lambda \iint \left(\breve{u}\tilde{u} + \breve{v}\tilde{v}\right) dxdy$$

After discretization, we obtain:

$$\tilde{A}\widetilde{w} = \lambda B \widetilde{w}$$

Complete program eigenadj.edp (look for ??? in this file) to compute the adjoint global modes.

6/ Compute the angle between the direct and adjoint global modes. Check bi-orthogonality of direct and adjoint global modes.

7/ Modify program eigen.edp to solve the eigen-problem:

$$A^* \widetilde{w}' = \mu B \widetilde{w}'$$

where A^* designates the transconjugate of matrix A. Compare $\widetilde{w}'$ and $\widetilde{w}$.

Show that: $(\mu^* - \lambda)\widetilde{w}'^* B \widehat{w} = 0$. Interpret the results.

8/ DNS simulations. We consider the Navier-Stokes equations in perturbative form: $w(t) = w_0 + w'(\text{t})$:

$$\begin{cases} \partial_t u' + u' \cdot \nabla u_0 + u_0 \cdot \nabla u' + u' \cdot \nabla u' & = & -\nabla p' + \nu \Delta u' \\ \nabla \cdot u' & = & 0 \end{cases}$$

A first –order semi-implicit discretization in time yields:

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\Delta t} + u^{n+1} \cdot \nabla u_0 + u_0 \cdot \nabla u^{n+1} + u^n \cdot \nabla u^n & = & -\nabla p^{n+1} + \nu \Delta u^{n+1} \\ \nabla \cdot u^{n+1} & = & 0 \end{cases}$$

This may be re-arranged into:

$$\begin{cases} \dfrac{u^{n+1}}{\Delta t} + u^{n+1} \cdot \nabla u_0 + u_0 \cdot \nabla u^{n+1} + \nabla p^{n+1} - \nu \Delta u^{n+1} & = & \dfrac{u^n}{\Delta t} - u^n \cdot \nabla u^n \\ \nabla \cdot u^{n+1} & = & 0 \end{cases}$$

Show that the weak form with $\breve{w}$ as the test-function is:

$$\iint \left(\breve{u}\left(\frac{u^{n+1}}{\Delta t} + u^{n+1}\,\partial_x u_0 + v^{n+1}\partial_y u_0 + u_0\partial_x u^{n+1} + v_0\partial_y u^{n+1}\right) - (\partial_x \breve{u})p^{n+1} + \nu\left(\partial_x\breve{u}\partial_x u^{n+1} + \partial_y\breve{u}\partial_y u^{n+1}\right) \right.$$
$$+ \breve{v}\left(\frac{v^{n+1}}{\Delta t} + u^{n+1}\partial_x v_0 + v^{n+1}\partial_y v_0 + u_0\partial_x v^{n+1} + v_0\partial_y v^{n+1}\right) - \left(\partial_y\breve{v}\right)p^{n+1}$$
$$\left. + \nu\left(\partial_x\breve{v}\partial_x v^{n+1} + \partial_y\breve{v}\partial_y v^{n+1}\right) + \breve{p}\left(\partial_x u^{n+1} + \partial_y v^{n+1}\right)\right) dxdy$$
$$= \iint \left(\frac{\breve{u}u^n}{\Delta t} - \breve{u}(u^n\cdot\nabla u^n) + \frac{\breve{v}v^n}{\Delta t} - \breve{v}(v^n\cdot\nabla v^n)\right) dxdy$$

After spatial discretization, we obtain:

$$Aw^{n+1} = b$$

In folder DNS,

FreeFem++ init.edp // Initial condition = real part of unit energy eigenvector in ../Eigs
FreeFem++ dns.edp // Launch linearized DNS simulation
Octave plotlinlog('out_0.txt',1,2,1) // plot perturbation energy as a function of time
Octave plotlinlin('out_0.txt',1,3,1) // plot u-velocity in wake as a function of time
FreeFem++ plotUvvp.edp // Plot flowfield after 100 time steps

9/ Perform a linearized DNS simulation with a unit energy adjoint flowfield as initial condition. Compare perturbation energy as a function of time with results obtained in 8/ Relate this result to the angle computed in 6/

10/ Perform a non-linear simulation to observe saturation (switch NL to 1 in dns.edp).

11/ Vary the Reynolds number, find critical Reynolds number with stability analyses and observe saturation amplitudes with non-linear simulations as a function of Reynolds number in the range $40 < Re < 100$.

Worksheet n°2: Multiple time-scale analysis and amplitude equations

Codes may be downloaded on: http://denissipp.free.fr/teaching/Codes/NumStud3.zip

1/ Direct numerical simulation of cylinder flow at Re=100

We solve the unsteady Navier-Stokes equations in perturbative form ($w \coloneqq w_0 + w$) around a cylinder flow at $Re = \nu^{-1} = 100$. The initial condition is the real part of a small amplitude unstable global mode.

$$\mathcal{B}\partial_t w + \mathcal{N}_{w_0} w + \mathcal{L}w = -\frac{1}{2}\mathcal{N}(w,w)$$

$$w(0) = \alpha \mathrm{Re}(\hat{w})$$

with :

$$w = \begin{pmatrix} u \\ v \\ p \end{pmatrix},\ \mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ \mathcal{N}(w_1, w_2) = \begin{pmatrix} u_1 \cdot \nabla u_2 + u_2 \cdot \nabla u_1 \\ 0 \end{pmatrix},\ \mathcal{N}_{w_0} w = \mathcal{N}(w_0, w),$$

$$\mathcal{L} = \begin{pmatrix} -\nu\Delta() & \nabla() \\ -\nabla \cdot () & 0 \end{pmatrix}$$

The base-flow and the global mode are defined by:

$$\frac{1}{2}\mathcal{N}(w_0, w_0) + \mathcal{L}w_0 = 0$$

$$\lambda \mathcal{B}\hat{w} + \left(\mathcal{N}_{w_0} + \mathcal{L}\right)\hat{w} = 0$$

In DNS/Mesh:

FreeFem++ mesh.edp

In DNS/BF:

FreeFem++ init.edp

FreeFem++ newton.edp

In DNS/Eigs:

FreeFem++ eigen.edp

In DNS/DNS:

FreeFem++ init.edp // generate initial condition from small amplitude global mode

FreeFem++ dns.edp // launch DNS simulation

Octave plotlinlog('out_0.txt',1,2,1) // represent energy as a function of time in fig 1

Octave plotlinlin('out_0.txt',1,4,2) // represent v velocity as a function of time

2/ Van der Pol Oscillator: multiple time-scale analysis

The Van der Pol Oscillator corresponds to the following governing equations:

$$w'' + \omega_0^2 w = 2\tilde{\delta} w' - w^2 w'$$

$$w(0) = w_I, w'(0) = 0$$

where the $(\cdot)'$ is the time-derivative, w_I is the initial condition, ω_0 the frequency and $\tilde{\delta}$ the instability strength. Here, we choose: $\omega_0 = 10$, $\tilde{\delta} = 0.3$ and $w_I = 0.01$.

2a/ Numerical time-integration

We integrate in time the above equations. For this,

In VanDerPol:

Octave pkg load all // load external packages for time integration, Fourier analysis, etc.

Octave vdp // integrate in time unforced Van der Pol equations

2b/ One time-scale approach

We try to approximate the solution by considering a small instability strength: $\tilde{\delta} = \delta\epsilon$, with $\epsilon \ll 1$ and $\delta = O(1)$. We look for an approximation of the solution with an expansion of the form:

$$w = \epsilon^{\frac{1}{2}} y \text{ and } y = y_0 + \epsilon y_1 + \cdots.$$

We first try with only one time-scale: $y(t) = y_0(t) + \epsilon y_1(t) + \cdots$

The second-order solution is given by:

$$w = \left(\tilde{A}e^{i\omega_0 t} + \text{c.c.}\right) + \left(\frac{-3\tilde{A}^3 + 12\tilde{\delta}\tilde{A}}{8\omega_0} ie^{i\omega_0 t} + \frac{\mathrm{i}\tilde{A}^3}{8\omega_0} e^{3i\omega_0 t} - \left(2\tilde{\delta}\tilde{A} - \tilde{A}^3\right)\left(\frac{1 + 2i\omega_0 t}{4\omega_0}\right) ie^{i\omega_0 t} + \text{c.c.}\right)$$

$$\tilde{A} = \frac{w_I}{2}$$

To represent this solution, in VanDerPol:

Octave clf // clear all figures

Octave vdp // integrate in time unforced Van der Pol equations

Octave vdp_tlr // show first and second order approximations with one time-scale

2c/ Two time-scales approach

The two time-scale first-order solution is given by:

$$w(t) = \left(\tilde{A}e^{i\omega_0 t} + \text{c.c.}\right)$$

with:

$$\frac{d\tilde{A}}{dt} = \tilde{\delta}\tilde{A} - \frac{1}{2}\tilde{A}^3$$

$$\tilde{A}(0) = \frac{w_I}{2}$$

To represent this solution, in VanDerPol:

Octave clf // clear all figures

Octave vdp // integrate in time unforced Van Der Pol equations

Octave vdp_tlr // show first and second order approximations with one time-scale

Octave vdp_mts // show first and second order approximations with two time-scales

3/ Van der Pol Oscillator with harmonic forcing

We consider the forced Van der Pol oscillator:

$$w'' + \omega_0^2 w = 2\tilde{\delta}w' - w^2 w' + \tilde{E}\cos\omega_f t,$$

where ω_f and $\tilde{E}$ are respectively the forcing frequency and the forcing amplitude. Here, we choose: $\omega_f = 25$ and $\tilde{E} = 600$. The first-order two time-scale solution is given by:

$$w(t) = 2\tilde{A}\cos(\omega_0 t + \phi) + \frac{\tilde{E}}{\omega_0^2 - \omega_f^2}\cos\omega_f t$$

with:

$$\frac{d\tilde{A}}{dt} = \left[\tilde{\delta} - \frac{1}{4}\left(\frac{\tilde{E}}{\omega_0^2 - \omega_f^2}\right)^2\right]\tilde{A} - \frac{1}{2}\tilde{A}^3$$

To represent this solution, in VanDerPol:

Octave clf // clear all figures

Octave vdpf // integrate in time unforced Van Der Pol equations

Vary the forcing amplitude $\tilde{E}$ from 0 to 600 and observe in each case the resulting frequency spectrum.

4/ Forced Navier-Stokes equations

We consider the Navier-Stokes equation in perturbative form ($w := w_0 + w$) with a forcing term acting on the momentum equations:

$$\mathcal{B}\partial_t w + \mathcal{N}_{w_0} w + \mathcal{L}w = \tilde{\delta}\mathcal{M}(w_0 + w) - \frac{1}{2}\mathcal{N}(w,w) + \left(\tilde{E}e^{i\omega_f t}f + \text{c.c}\right).$$

Here:

$$w = \begin{pmatrix} u \\ v \\ p \end{pmatrix},\ \mathcal{L} = \begin{pmatrix} -\nu_c\Delta() & \nabla() \\ -\nabla\cdot() & 0 \end{pmatrix},\ \mathcal{M} = \begin{pmatrix} -\Delta & 0 \\ 0 & 0 \end{pmatrix}.$$

The viscosity ν has been replaced by $\nu = \nu_c - \tilde{\delta}$, where ν_c is the critical viscosity which achieves marginal stability of the linear dynamics $Re_c = \nu_c^{-1} = 46.6$.

The base-flow is given by:

$$\frac{1}{2}\mathcal{N}(w_0, w_0) + \mathcal{L}w_0 = 0,$$

while $\tilde{E}$ and ω_f correspond respectively to the forcing amplitude and forcing frequency. The forcing structure f (acting solely on the momentum equations, so that $\mathcal{B}f = f$) is also given.

In the following, we consider a slightly supercritical regime (the Reynolds number is slightly above the critical Reynolds number):

$$\tilde{\delta} = \epsilon\delta,\ \epsilon \ll 1, \delta = O(1),$$

and a small- amplitude forcing, which scales as:

$$\tilde{E} = \epsilon^{\frac{1}{2}}E, E = O(1).$$

We look for an approximation of the solution under the form:

$$w = \epsilon^{\frac{1}{2}}\left(y_0(t, \tau = \epsilon t) + \epsilon^{\frac{1}{2}}y_{\frac{1}{2}}(t, \tau = \epsilon t) + \epsilon^1 y_1(t, \tau = \epsilon t) + \cdots\right)$$

The second-order solution is given by:

$$w = \left(\tilde{A}e^{i\omega_c t}y_A + \text{c.c}\right) + \left(\tilde{E}e^{i\omega_f t}y_E + \text{c.c}\right) + \tilde{\delta}w_\delta + \left(\tilde{A}^2 e^{2i\omega_c t}y_{AA} + \text{c.c.}\right) + \left|\tilde{A}\right|^2 y_{A\bar{A}} + \left|\tilde{E}\right|^2 y_{E\bar{E}} + \left(\tilde{A}\tilde{E}e^{i(\omega_c+\omega_f)t}y_{AE} + \text{c.c.}\right) + \left(\tilde{A}\bar{\tilde{E}}e^{i(\omega_c-\omega_f)t}y_{A\bar{E}} + \text{c.c.}\right) + \cdots$$

With :

$$i\omega_c\mathcal{B}y_A + \mathcal{N}_{w_0}y_A + \mathcal{L}y_A = 0$$

$$i\omega_f\mathcal{B}y_E + \mathcal{N}_{w_0}y_E + \mathcal{L}y_E = f$$

$$\mathcal{N}_{w_0}y_\delta + \mathcal{L}y_\delta = \mathcal{M}y_0$$

$$2i\omega_c\mathcal{B}y_{AA} + \mathcal{N}_{w_0}y_{AA} + \mathcal{L}y_{AA} = -\frac{1}{2}\mathcal{N}(y_A, y_A)$$

$$\mathcal{N}_{w_0}y_{A\bar{A}} + \mathcal{L}y_{A\bar{A}} = -\mathcal{N}(y_A, \bar{y}_A)$$

$$\mathcal{N}_{w_0}y_{E\bar{E}} + \mathcal{L}y_{E\bar{E}} = -\mathcal{N}(y_E, \bar{y}_E)$$

$$2i(\omega_c + \omega_f)\mathcal{B}y_{AE} + \mathcal{N}_{w_0}y_{AE} + \mathcal{L}y_{AE} = -\mathcal{N}(y_A, y_E)$$

$$2i(\omega_c - \omega_f)\mathcal{B}y_{A\bar{E}} + \mathcal{N}_{w_0}y_{A\bar{E}} + \mathcal{L}y_{A\bar{E}} = -\mathcal{N}(y_A, \bar{y}_E)$$

And:

$$\frac{d\tilde{A}}{dt} = \lambda\tilde{\delta}\tilde{A} - \mu\tilde{A}\left|\tilde{A}\right|^2 - \pi\tilde{A}\left|\tilde{E}\right|^2$$

where:

$$\lambda = < \tilde{y}_A, \mathcal{M}y_A > - < \tilde{y}_A, \mathcal{N}(y_A, y_\delta)$$

$$\mu = < \tilde{y}_A, \mathcal{N}(y_A, y_{A\bar{A}}) + \mathcal{N}(\bar{y}_A, y_{AA}) >$$

$$\pi = < \tilde{y}_A, \mathcal{N}(y_A, y_{E\bar{E}}) + \mathcal{N}(y_{\bar{E}}, y_{AE}) + \mathcal{N}(y_E, y_{A\bar{E}}) >$$

$$-i\omega_c\mathcal{B}\tilde{y}_A + \tilde{\mathcal{N}}_{w_0}\tilde{y}_A + \tilde{\mathcal{L}}\tilde{y}_A = 0$$

$$< \tilde{y}_A, \mathcal{B}y_A > \; = 1$$

4a/ In AmplEq/Mesh:

FreeFem++ mesh.edp // generate mesh

In AmplEq/BF:

FreeFem++ init.edp // generate initial guess for Newton iterations

FreeFem++ newton.edp // Newton iteration

In AmplEq/Eigs:

FreeFem++ eigen.edp // compute global mode

FreeFem++ eigenadj.edp // compute adjoint global mode

FreeFem++ norm.edp // generate scaled adjoint global mode

In AmplEq/WNL:

FreeFem++ udelta.edp // generate modification of base-flow due to increase in Reynolds number

FreeFem++ uAA.edp // generate second harmonic due to interaction of global mode with himself

FreeFem++ uAAb.edp // generate zero-harmonic due to interaction of global mode with adjoint of himself

FreeFem++ lambda.edp // compute λ coefficient of Stuart-Landau equation

FreeFem++ mu.edp // compute μ coefficient of Suart-Landau equation

FreeFem++ forcing.edp // define external forcing (spatial structure anf frequency)

FreeFem++ uE.edp // compute response due to external forcing

FreeFem++ uAE.edp // compute AE-harmonic due to interaction of response to external forcing with global mode

4b/ Complete program uAEb.edp to compute the $A\bar{E}$ harmonic due to the interaction of the global mode with the adjoint of the response due to external forcing.

4c/ Complete program uEEb.edp to compute the zero-harmonic due to the interaction of the external forcing response with the conjugate of himself.

4d/ Complete program pi.edp to compute the π coefficient.

5/ Forced Direct numerical simulation

We integrate in time the forced Navier-Stokes equations at $Re = \nu^{-1} = 100$:

$$\mathcal{B}\partial_t w + \mathcal{N}_{w_0}w + \mathcal{L}w = -\frac{1}{2}\mathcal{N}(w, w) + \left(\tilde{E}e^{i\omega_f t}f + \text{c.c}\right)$$

where:

$$w = \begin{pmatrix} u \\ v \\ p \end{pmatrix}, \; \mathcal{L} = \begin{pmatrix} -\nu\,\Delta() & \nabla() \\ -\nabla\cdot() & 0 \end{pmatrix}$$

In DNS/DNS:

FreeFem++ dnsf.edp // launch forced DNS simulation

Octave plotlinlog('out_4000.txt',1,2,1) // represent energy as a function of time in fig 1

Octave plotlinlin('out_4000.txt',1,4,2) // represent v velocity as a function of time in fig 2

Octave spectrum // compare spectrum with and without control